Homework # 1

Due Wednesday, January 18, 2006, at 1:30 PM

Collaboration is allowed and encouraged

Definition of a Boolean algebra (a reminder): An algebraic system consisting of a set of elements B, where B has at least two elements 0 and 1 and two binary operations + and $\cdot$, is a Boolean algebra if the following four axioms hold:

- **A1. Identities:** $a + 0 = a$ and $a \cdot 1 = a$
- **A2. Complements:** $a + \bar{a} = 1$ and $a \cdot \bar{a} = 0$
- **A3. Commutativity:** $a + b = b + a$ and $a \cdot b = b \cdot a$
- **A4. Distributivity:** $a + b \cdot c = (a + b) \cdot (a + c)$ and $a \cdot (b + c) = a \cdot b + a \cdot c$

In the following problems you will enjoy proving some fun properties of Boolean algebras. You can use the aforementioned axioms as well the following properties that we proved in class:

- **T0. Duality:**
  Correctness of a Boolean identity is maintained when interchanging + and $\cdot$, as well as 0 and 1.
- **T1. Distinct Complement:**
  Every element has another element that is its unique complement.
- **T2. Absorption:**
  $a + a \cdot b = a$ and $a \cdot (a + b) = a$
- **L1. Self Absorption:**
  $a + a = a$ and $a \cdot a = a$
- **L2: Simple Absorption:**
  $a + 1 = 1$ and $a \cdot 0 = 0$

In the proofs *please justify each step* by referring to the fact you are using.

1. **Simple Properties of Boolean Algebras**

   (a) Prove that 0 and 1 are complements. Namely, $\bar{0} = 1$ and $\bar{1} = 0$.

   (b) Prove that $\overline{(\bar{a})} = a$.

   (c) Prove the following identity: $a + \bar{a} \cdot b = a + b$

2. **Boolean Integers**
   Let n be the product of distinct prime numbers. Let BI_n be the set of all divisors of n. For example, for $n = 30 = 2 \times 3 \times 5$ we have $BI_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

   Consider the two binary operations *lcm* (Lowest Common Multiple) and *gcd* (Greatest Common Divisor).

   Prove that the algebraic system consisting of BI_n (for every n that is a product of distinct primes), the operations *lcm* and *gcd* with the 0 element being the integer 1 and the 1 element being the integer n is a Boolean algebra. Note that you also need to prove what is the complement.

3. **Associativity**
   In this problem you will prove that the two operations of a Boolean algebra are associative, namely:
   $$(a + b) + c = a + (b + c)$$
   and
   $$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
   Prove the following statements and reach the conclusion.

   (a) Prove that:
   $$a + a \cdot (b \cdot c) = a + (a \cdot b) \cdot c$$

   (b) Prove that:
   $$\bar{a} + a \cdot (b \cdot c) = \bar{a} + (a \cdot b) \cdot c$$

   (c) Prove that:
   $$(a + a \cdot (b \cdot c)) \cdot (\bar{a} + a \cdot (b \cdot c)) = a \cdot (b \cdot c)$$
   and
   $$(a + (a \cdot b) \cdot c) \cdot (\bar{a} + (a \cdot b) \cdot c) = (a \cdot b) \cdot c$$

   (d) Based on (a)-(c) above argue that you proved Associativity.

   (e) *Extra credit:* Propose a different (simpler?) proof for Associativity.

4. **DeMorgan Laws**

    (a) Prove that

    $$\overline{(a+b)} = \bar{a} \cdot \bar{b}$$

    and

    $$\overline{(a \cdot b)} = \bar{a} + \bar{b}$$

    (b) Find the complement of $a \cdot \bar{b} + \bar{a} \cdot b$.

    (c) Prove that $\overline{(a+b+c)} = \bar{a} \cdot \bar{b} \cdot \bar{c}$.

5. **Equality Properties**

    (a) Let $a + b = 0$, can you conclude that both $a = 0$ and $b = 0$?

    (b) Let $a + b = 1$, can you conclude that either $a = 1$ or $b = 1$?

    (c) Prove that $a = b$ if and only if $a \cdot \bar{b} + \bar{a} \cdot b = 0$.