Homework # 3

Due Wednesday, February 8, 2006, at 1:30 PM

Collaboration and/or discussion are not allowed on this homework set

1. **Arbitrary Boolean Symbols**
In this problem we study the representation of the Boolean variables by values which are not 0 and 1. Note that throughout this problem we will assume that the values of the output of a linear threshold function are still either 0 or 1.

   (a) Assume that the Boolean variables are represented by 1's and -1's via the mapping $0 \rightarrow 1$ and $1 \rightarrow -1$. For example,
   $$AND(x_1, x_2, \cdots, x_n) = \begin{cases} 1 & \text{if the number of -1's in } X \text{ is } n \\ 0 & \text{otherwise} \end{cases}$$
   and
   $$OR(x_1, x_2, \cdots, x_n) = \begin{cases} 0 & \text{if the number of 1's in } X \text{ is } n \\ 1 & \text{otherwise} \end{cases}$$
   Derive the linear threshold representation of $AND(x_1, x_2, \cdots, x_n)$ and $OR(x_1, x_2, \cdots, x_n)$, for $X \in \{1, -1\}^n$.

   (b) What is the linear threshold representation of $AND(\bar{x}_1, x_2, \bar{x}_3, x_4, x_5)$ (for $X \in \{1, -1\}^5$)?

   (c) Let
   $$f(X) = sgn(w_0 + \sum_{i=1}^{n} w_i x_i)$$
   be the linear threshold representation of a Boolean function $f(X)$ using the variables 0 and 1.
   We want to change the input variables into the real numbers a and b via the mapping $0 \rightarrow a$ and $1 \rightarrow b$, without changing the output f, namely, $f(X) \in \{0, 1\}$.
   Derive the linear threshold representation of $f(X)$ for $X \in \{a, b\}^n$.

   (d) Let $X \in \{a, b\}^n$ and let
   $$XOR(x_1, x_2, \cdots, x_n) = \begin{cases} 1 & \text{if the number of b's in } X \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$
   Prove that, for arbitrary a and b, $n \geq 2$, $a \neq b$, there is no linear threshold representation for $XOR(x_1, x_2, \cdots, x_n)$.

2. **Multilevel Threshold Functions**
In this problem we explore functions with nonbinary inputs. Assume that the inputs $x_i \in \{0, \ldots, q-1\}$ (the binary case is when $q = 2$).

We define

$$OR_q(x_1, x_2, \ldots, x_n) = \begin{cases} 0 & \text{if } x_i = 0 \text{ for all } 1 \le i \le n \\ 1 & \text{otherwise} \end{cases}$$

and

$$AND_q(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } x_i \neq 0 \text{ for all } 1 \le i \le n \\ 0 & \text{otherwise} \end{cases}$$

   (a) For which $q \ge 2$ $OR_q \in LT_1$?

   (b) For which $q \ge 2$ $AND_q \in LT_1$?

   (c) Characterize the set of functions g for which the function

$$f_q(x_1, x_2, \ldots, x_n) = g(y_1, y_2, \ldots, y_n)$$

   with

$$y_i = \begin{cases} 0 & \text{if } x_i = 0 \\ 1 & \text{otherwise} \end{cases}$$

   satisfies $f_q \in LT_1$ for all $q \ge 2$.

   Note: You can see that in part (a), g was OR, and in part (b), g was AND.

3. **The Complete Quadratic Function**
The Complete Quadratic (CQ) function is the Boolean function that consists of the XOR of all the $\binom{n}{2}$ possible AND's between pairs of variables. Namely,

$$CQ(X) = (x_1 \wedge x_2) \oplus (x_1 \wedge x_3) \oplus \cdots \oplus (x_{n-1} \wedge x_n).$$

For example,

$$CQ(x_1, x_2, x_3) = (x_1 \wedge x_2) \oplus (x_1 \wedge x_3) \oplus (x_2 \wedge x_3).$$

   (a) Prove that $CQ(X)$ is a symmetric function. Express $CQ(X)$ as a function of $|X|$.

   (b) For which n (number of inputs), $CQ(X) \notin LT_1$? Prove your claim.

   (c) Draw the depth-2 *layered* LT circuit that computes $CQ(X)$ with $X \in \{0,1\}^5$.

   (d) Prove that $CQ(x_1, x_2, \cdots, x_n)$ with $n = 2^k - 1$, can be implemented by an LT circuit with $k - 1$ gates. Draw the circuits for the cases $k = 2$ and $k = 3$.