Homework # 4

Due Wednesday, February 22, 2006, at 1:30 PM
Collaboration is allowed and encouraged

1. **Generalizing the Non-layered Construction**
A symmetric function $f(x_1, x_2, \cdots, x_n)$ can be described by a vector of length $n+1$ that we call the symmetric function table and denote it by $V(f) = (v_0, v_1, \cdots v_n)$, where

$$v_i = \begin{cases} 1 & \text{if for } |X| = i \quad f(X) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $|V(f)|$ be the number of 1's in $V(f)$. In class we described a method for constructing a depth-2 *non-layered* LT circuit with $|V(f)|$ +1 gates.

   (a) Let $V(f) = \{011001010110\}$ be the symmetric function table of f (a function of 11 variables). Using the approach from class we can construct a depth-2 *non-layered* LT circuit with 7 gates ($|V(f)| = 6$). However, note that $V(f)$ has only 4 intervals of 1's. Show how to implement f with a depth-2 *non-layered* LT circuit with 5 gates.

   (b) Prove the general result. Namely, assume that you are given a symmetric function f with $V(f)$ consisting of k intervals of 1's. Prove that f can be implemented by a depth-2 *non-layered* circuit with $k+1$ gates.

Strong hint: Look at the paper "Linear-Input Logic", by R. C. Minnick, that is posted on the class web site.

2. **Computing the Spectrum**
Let $f_1(x_1, x_2) = x_1 \wedge x_2$ (AND function of two variables) and $f_2(x_1, x_2) = x_1 \vee x_2$ (OR function of two variables). In class we proved that AND and OR have the following polynomial representation.

$$f_1(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1 x_2)$$

$$f_2(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1 x_2)$$

(a) Derive the polynomial representation of $f_3(x_1, x_2, x_3) = x_1 \wedge (x_2 \vee x_3)$.

(b) Derive the polynomial representations of:

$$AND(x_1, x_2, x_3) = x_1 \wedge x_2 \wedge x_3$$

$$OR(x_1, x_2, x_3) = x_1 \vee x_2 \vee x_3$$

(c) In general, for an arbitrary n, compute the coefficients of the polynomial representations of $AND(x_1, x_2, \ldots, x_n)$ and $OR(x_1, x_2, \ldots, x_n)$ (note that the coefficients are functions of n).

3. **The Spectrum of Symmetric Functions**
A Boolean function is symmetric if and only if it is a function of the number of 1's in the input. For example, PARITY, AND and OR are symmetric functions. Notice that the degree of their polynomial representation is n.

(a) Prove that the degree of the polynomial representation of an arbitrary symmetric function f of n variables is at least $\lceil n/2 \rceil$.

(b) For every n *odd*, find a symmetric function with n variables such that the degree of its polynomial representation is $n - 1$.

(c) *Extra credit (15%):* Find an infinite sequence of *even* numbers, such that for every number n in the sequence there is a symmetric function with n variables such that the degree of its polynomial representation is $n - 1$.

4. **The $\{0, 1\}$ Representation**
In class we proved that every Boolean function $f(X) \in \{1, -1\}$, $X \in \{1, -1\}^n$, can be uniquely expressed as a polynomial with rational coefficients. The coefficients of the polynomial representation can be computed using the Sylvester-type Hadamard matrix.

In this problem we assume that we use $\{0, 1\}$, namely a Boolean function $f(X) \in \{0, 1\}$ is defined using $X \in \{0, 1\}^n$ and study the corresponding polynomial representation that we call the $\{0, 1\}$-polynomial representation. For example, the $\{0, 1\}$-polynomial representation of $AND(x_1, x_2)$ is $x_1 x_2$.

(a) Derive the $\{0, 1\}$-polynomial representation of $OR(x_1, x_2, x_3)$ and $XOR(x_1, x_2, x_3)$.

(b) In general, prove that the $\{0, 1\}$-polynomial representation is unique.

(c) Computing the coefficients: What is the transformation matrix from the function to the coefficients of the $\{0, 1\}$-polynomial representation?

Medium hint: derive the recursive definition of the transformation matrix.