Homework # 1

Due Tuesday April 15, 2008, at 2:30 PM

Collaboration is allowed and encouraged

1. **Fingers**

   It is well known that our fingers can be used to add two numbers (in this problem we consider only integers). A less known fact is that our fingers can be used to multiply numbers. Assume that you know the multiplication table from 1×1 to 5×5 and that you want to multiply two numbers k and l, where $5 < k \leq 10$ and $5 < l \leq 10$. The following example demonstrates the idea of this ancient method for multiplication.

   Consider the multiplication 7×9: Bend two fingers in the left hand - that corresponds to the excess over 5 of the multiplier. Bend four fingers in right hand - that corresponds to the excess over 5 of the multiplicand.

   Count the bent fingers: $2 + 4 = 6$. These will be the tens of the final result. Multiply the standing fingers: $3 \times 1 = 3$. These will be the units of the final result. Namely the answer is: 63. Hence, knowledge of the addition and the multiplication tables up to 5 is sufficient.

   (a) Try the procedure for 8×8, 6×8 and 6×7. Describe the steps. You will notice that you need to revise the procedure for it to work. Formalize the general method for finger multiplication for numbers k and l, where, $5 < k \leq 10$ and $5 < l \leq 10$.

   (b) Prove analytically (not exhaustively) that your proposed method from (a) works.

   (c) Suggest a similar finger multiplication method for numbers k and l, such that $10 \leq k \leq 15$ and $10 \leq l \leq 15$. Prove analytically (not exhaustively) that your proposed method works.

   (d) Suppose that we are endowed with six fingers in each hand. Suggest and prove correctness of a corresponding finger multiplication method (consider the analogous cases to both (a) and (c)).
   Hint: having six fingers leads to a base 12 number system.

   (e) What is the corresponding general method for finger multiplication for the case of m fingers in each hand? (consider the analogous cases to both (a) and (c)).

2. **Coin Systems**

This problem is inspired by the ancient method of using tokens for representing quantities. Interestingly, even today, the token method is the predominant method for representing money (credit cards are a different story...).

Part 1

You were hired by the local super-market to work as a cashier. This particular super-market, as a policy, only returns coins as change. The cashier has an unlimited supply of the following coins: 1¢, 5¢, 10¢, 25¢. The job requires a cashier to always return change using the *minimal* possible number of coins.

(a) Return change of 114¢. Explain your calculations. You do not need to prove the minimality of your coin set (in Part 1 only, non-minimal solutions get full credit as well.).

(b) Return change of 13213¢. Explain your calculations. You do not need to prove the minimality of your coin set (in Part 1 only, non-minimal solutions get full credit as well.).

Part 2

Following an abundance of Caltech students cashiers, the management decided to use a more complex method based on proprietary coins with the following denominations: 1¢, 5¢, 18¢, 29¢.

(c) Return change of 114¢. Prove that your solution is minimal.

(d) Return change of 16225¢. Prove that your solution is minimal. *Hint: for any change value y, the minimal coin set cannot be smaller than y divided by the largest available coin.*

Part 3

Consider the following coin system: 1¢, 2¢, 4¢, 7¢, 13¢, 24¢.

(e) Return two different *minimal* coin sets for the change value of 34¢. Prove that the two solutions are *minimal*.

(f) Prove that there is no *minimal* coin set that contains all 6 coin denominations.

(g) Prove that, except for even multiples of 24¢, for any change value *there exists a minimal* coin set with at least one coin returned with an odd multiplicity.

Part 4

For the US official coin system (the one in Part 1), we calculate change by repeatedly taking the largest coin less than or equal to the amount remaining. In this part you will prove that, for the official US system, this method is guaranteed to give a minimal number of coins for any change value. In fact, you will prove a more general result.

Definitions:

(1) In an arbitrary coin system with m coins let $c_1, \ldots, c_m$ be the values of the coins with $c_i < c_{i+1}$ for $1 \leq i < m$. For example, for the coins in Part 1, $m = 4$ and $c_1 = 1$, $c_2 = 5$, $c_3 = 10$, $c_4 = 25$.

(2) For any change value x, and a given coin system, assume that the change x is obtained by a *greedy procedure,* namely, repeatedly taking the largest coin less than or equal to the amount remaining. We define $G(x)$ to be the number of coins that are obtained for a change x using the greedy procedure.

(3) For any change value x, and a given coin system, define $M(x)$ to be the number of coins in a minimal set of coins that sum to x.

Note that by the definitions: $M(x) \leq G(x)$.

(h) Given an arbitrary coin system with m coins, with $x \geq c_m$; prove that $G(x) = G(x - c_m) + 1$.

(i) Let c_j be an arbitrary coin in a coin system with m coins, with $x \geq c_j$; prove that $M(x) \leq M(x - c_j) + 1$. For which x's and c_j's we have that $M(x) = M(x - c_j) + 1$? Give an example of a coin system with c_k, c_l and x such that $M(x) = M(x - c_k) + 1$, and $M(x) < M(x - c_l) + 1$.

(j) Given an arbitrary coin system with m coins. Assume that $x \geq c_{m-1} + c_m$. Assume the following inductive hypothesis: $G(y) = M(y)$ for all $y < x$.

Assume further that the coin c_i, $1 \leq i < m$, is in a minimal coin set for x. Use the inductive hypothesis to prove that $G(x) \leq G(x - c_m - c_i) + 2$.

Hint: use the facts you proved in (h) and (i).

(k) Prove the conclusion (it also provides an algorithm for checking if the *greedy procedure* is optimal for a given coin system). Given a coin system with m coins, such that $M(y) = G(y)$ for any $y < c_{m-1} + c_m$. Prove that $M(x) = G(x)$ for any x by justifying the following steps. In the following assume that c_i, $1 \leq i < m$, is a coin in the minimal solution.

$$\begin{aligned} G(x) &\leq G(x - c_m - c_i) + 2 \\ &= G(x - c_i) + 1 \\ &= M(x - c_i) + 1 \\ &= M(x) \\ &\leq G(x) \end{aligned}$$

(l) Apply (k) from above to prove that the greedy procedure for calculating change for the official US coin system provides the minimal solution.

(m) Prove or disprove the statement that $M(x) = G(x)$ for the coin system in Part 2, namely, $c_1 = 1¢$, $c_2 = 5¢$, $c_3 = 18¢$, $c_4 = 29¢$.