

Homework # 2

Due Tuesday, April 22, 2008, at 2:30 PM
Collaboration is allowed and encouraged

1. Roman Numerals

In this problem we study and compare three variations of Roman numerals representation systems. Throughout the problem we consider the following symbols: M, D, C, L, X, V, I (with values 1000, 500, 100, 50, 10, 5 and 1, respectively).

UR: Uncompressed Roman Numerals.

This is a basic weighted representation system with symbols written in non-increasing order, from left to right. Each symbol adds its value to the total number. Only shortest representations are legal - namely, we apply the compression rules from class. Examples: *IIIII* is illegal, and should be represented by *V*; $49 = XXXXVIII$ and $56 = LVI$ are legal UR representations.

SR: Standard Roman Numerals.

Similar to the uncompressed method, only with some restricted compression. Symbols are written in non-increasing order, from left to right, except for when the following allowed changes *result in a shorter representation*.

Only the *I*, *X*, and *C* symbols can be used out of order for compression. Each compression symbol can appear only to the left of the two consecutive larger symbols. When an *I*, *X* and *C* appear out of order we subtract 1, 10 and 100, respectively. Namely *I* can appear on the left of *V* or *X*; *X* can appear to the left of *L* or *C*; and *C* can appear to the left of *D* or *M*. For example, $IX = 9$ is a legal *SR* representation, and *VL* and *IC* are illegal *SR* representations.

A symbol can be used *only once* for compression, and should appear in the right most location that provides the shortest representation.

Examples: *IIX* is illegal because *I* appears twice. *IXL* is illegal because *I* appears to the left of *L*. *IXIV* is illegal because *I* appears twice. *IXV* is illegal because it is not shorter than *XIV*. *XCIX* is legal because *X* appears only once for compression. *IXI* is illegal because it is not the shortest representation.

FC: Fully Compressed Roman Numerals.

A more flexible system with the following simple rule. If a symbol appears anywhere to the left of a symbol with greater value, then its value is *subtracted* from the total number. Otherwise it is added. For example: $IL = 49$ and $IXXC = 79$. An FC representation is legal only if there is no shorter FC representation, namely, a legal FC representation is minimal. For example, $XXXC = 70$ is illegal because $LXX = 70$ is shorter. Notice that in the FC system a value can have multiple legal representations.

- (a) We would like to get an idea about the descriptive power of each of the three representations. For that we will compare the number of symbols required to represent selected numbers between 1 and 100. Complete the following table. For each row, in columns UR , SR , FC , place a legal representation for x in the respective representation systems. In the rightmost column, fill the tuple $(\#UR, \#SR, \#FC)$ by recording the number of symbols in each representation from your solutions.

x	UR	SR	FC	$(\#UR, \#SR, \#FC)$
12				(, ,)
14				(, ,)
36				(, ,)
44				(, ,)
76				(, ,)
88				(, ,)
97				(, ,)
99				(, ,)

- (b) What are the values of the following FC numbers: (1) $IIIIC$, (2) $VLDI$ and (3) ICV . Are the three numbers legal in the FC system? How will you represent these values legally using the UR , SR and FC systems?
- (c) What is the smallest number for which the FC system requires a smaller number of symbols than the SR system? Justify your solution.
What are the numbers up to 50 for which the FC system requires a smaller number of symbols than the SR system?
- (d) For each of the three systems (UR , SR and FC) find a number between 1 and 500 that requires the maximal number of symbols. Prove that your solutions are correct.

2. Radix- b no-0 Positional Systems

In class we studied the radix-10 no-0 positional system where a positive integer N is represented as follows:

$$N = \sum_{i=0}^m d_i 10^i$$

with

$$d_i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, A = 10\}$$

for all $0 \leq i \leq m$.

Examples, $N = 10 = A$, $N = 100 = 9A$ and $N = 0 =$ (the blank string).

- (a) Compare $AAAA$ with 11111 , which number is bigger?

Does the domination principle hold for the radix-10 no-0 system? Prove your claim.

Recall that the domination principle for positional systems is: a number x is bigger than a number y if in the most significant position where x and y differ - the corresponding digit in x is bigger than the corresponding digit in y .

- (b) What is the radix-10 no-0 representation of 2110002105 ? What is the radix-10 representation of $1AA92A99A$.

Devise and describe an efficient procedure (syntax based) that transforms a base-10 representation to a base-10 no-0 representation. Devise and describe the inverse procedure (syntax based) that transforms a base-10 no-0 representation to a base-10 representation. In both cases argue about the correctness and performance of your procedure.

- (c) Suggest an analogous base-2 no-0 representation.

Hint: the number 4 is represented by 100 in base-2 and by 12 is base-2 no-0 representation.

Represent the base-2 number 100001 in base-2 no-0 representation. Represent the base-2 no-0 number 11222111122 in base-2 representation.