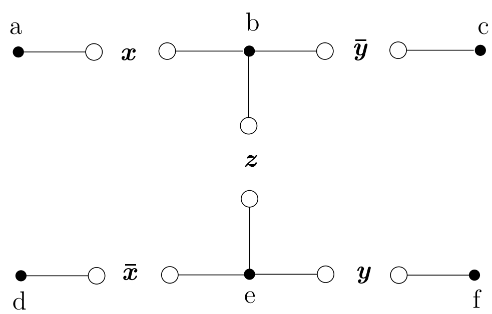
Homework #4

Due Thursday, May 25, 2017, at 2:30 PM
Collaboration and discussions are not allowed on Problem 1
and are allowed and encouraged on Problem 2

This homework set is based on *C. E. Shannon's Masters Thesis from 1938, "A Symbolic Analysis of Relay and Switching Circuits"*. This classical paper was the first work that connected between logic and circuit design. We encourage you to read this thesis, it is posted on the class web site.
Please note that Shannon used the dual notation, however, in this homework set we expect you to use the notation from class, namely, 0 corresponds to an open relay and 1 to a closed relay.

1. **A Generalization to Multiple Terminals** (*Collaboration is not allowed on this Problem*)
A generalization of the switching (relay) circuit model is a circuit with multiple terminals. The Boolean function X_{ab} is 1 if there is a closed path between terminals a and b, and 0 otherwise. With multiple terminals, $a, b, c, d, \ldots$, Boolean functions exist between any *pair* of terminals. For example, for the circuit



we have

$$X_{af} = x \cdot y \cdot z$$
$$X_{bd} = \bar{x} \cdot z$$
$$X_{cf} = 0$$

and so on.

(a) Construct a circuit with 3 relays that implements the functions

$$\begin{aligned} f_1 &= x \cdot y \\ f_2 &= \bar{x} \cdot y \end{aligned}$$

(b) Construct a circuit with 4 relays that implements the functions

$$\begin{aligned} f_1 &= x \cdot y + \bar{x} \cdot \bar{y} \\ f_2 &= \bar{x} \cdot y + x \cdot \bar{y} \end{aligned}$$

(c) Construct a circuit with 6 relays that implements the functions:

$$\begin{aligned} f_1 &= x \cdot (y + z) \\ f_2 &= y \cdot (x + z) \\ f_3 &= z \cdot (x + y) \\ f_4 &= x + y \cdot z \\ f_5 &= y + x \cdot z \\ f_6 &= z + x \cdot y \end{aligned}$$

(d) *Extra Credit: 10% of the total homework grade.* Construct a circuit with as few relays as possible that implements all 16 functions of two variables. A solution with 8 relays gets full credit.

2. **The Complete Quadratic Function** (*Collaboration <u>is allowed</u> on this Problem*)
The Complete Quadratic (CQ) function is the Boolean function that consists of the XOR of all the $\binom{n}{2}$ possible AND's between pairs of inputs. Namely,

$$CQ(X) = (x_1 \cdot x_2) \oplus (x_1 \cdot x_3) \oplus \cdots \oplus (x_{n-1} \cdot x_n).$$

For example,

$$CQ(x_1, x_2, x_3) = (x_1 \cdot x_2) \oplus (x_1 \cdot x_3) \oplus (x_2 \cdot x_3).$$

(a) In class we proved that a Boolean function $f(X)$ is symmetric iff it is a function of $|X|$ (the number of 1s in X). Write $CQ(X)$ with 6 inputs as a function of $|X|$. Note that there are 7 entries in the table.

(b) Generalization: For an arbitrary n, express $CQ(X)$ as a function of $|X|$. Namely, you need to specify, as a mathematical expression, the values of $|X|$ for which $CQ(X) = 1$. Justify your solution.

(c) Construct a circuit for $CQ(X)$ with five inputs, namely, $X \in \{0, 1\}^5$. Please use as few relays as possible. A solution with 20 relays (or less) gets full credit. Justify your solution.
Hint: Use the construction of XOR from class as an inspiration.