Homework #3

Due Tuesday, May 22, 2018, at 2:30 PM

Collaboration and discussions are not allowed on Problem 1
and are allowed and encouraged on Problem 2

1. **Syntax Boxes and the Binary Adder** (*Collaboration is not allowed on this Problem*)
In class we showed that the following syntax box $m(a, b)$ is magical, namely, a composition of m-boxes (called a circuit) can compute an arbitrary n-input binary syntax box.

$m(a, b) =$

a	b	m
0	0	1
0	1	1
1	0	1
1	1	0

(a) The *Parity* of two variables is defined by the following syntax box.

$Parity(a, b) =$

a	b	$Parity$
0	0	0
0	1	1
1	0	1
1	1	0

You need to design a circuit of m-boxes to compute $Parity(a, b)$. Your goal is to construct a circuit that consists of the smallest number of m-boxes possible. Please show your work and draw the circuit. An answer with 5 m-boxes or less will get full credit.

The binary adder has three inputs and two outputs. The outputs are the binary representation of the number of 1s in the inputs. Specifically, we showed in class that the first output (the sum) is $Parity(a, b, c)$ and the second output (the carry) is $Majority(a, b, c)$.

$Parity(a, b, c)$ =

a	b	c	$parity$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$Majority(a, b, c)$ =

a	b	c	$majority$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

(b) You need to design a circuit of m-boxes to compute $Parity(a, b, c)$. Your goal is to construct a circuit that consists of the smallest number of m-boxes possible. Please show your work and draw the circuit. An answer with 9 m-boxes or less will get full credit.

(c) You need to design a circuit of m-boxes to compute $Majority(a, b, c)$. Your goal is to construct a circuit that consists of the smallest number of m-boxes possible. Please show your work and draw the circuit. An answer with 8 m-boxes or less will get full credit.

2. **Boolean Proofs** (*Collaboration is allowed on this Problem*)
Here you will complete the proof of the theorems that were presented in class. *Please justify every step in your proofs using the axioms, lemmas and theorems from class.* However, there is one exception, you *cannot use the 0-1 Theorem* we proved in class.

Associativity Theorem
Here is the Associativity Theorem as was stated in class.
For any three elements a, b and c in a Boolean algebra the following is true:

$$(a + b) + c = a + (b + c)$$

and the dual statement:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

(a) The proof in class had 4 parts, however, we did not prove part (2). Prove the following statement (part (2) from class):

$$\bar{a} + (a \cdot (b \cdot c)) = \bar{a} + ((a \cdot b) \cdot c)$$

DeMorgan's Theorem
Here is DeMorgan's Theorem as was stated in class.
For any two elements a and b in a Boolean algebra the following is true:

$$\overline{(a \cdot b)} = \bar{a} + \bar{b}$$

and the dual statement:

$$\overline{(a + b)} = \bar{a} \cdot \bar{b}$$

The idea in the proof is to show that $(\bar{a} + \bar{b})$ is the complement of $(a \cdot b)$. Namely, by axiom **A2** we need to prove the following two statements:

(I) $(a \cdot b) + (\bar{a} + \bar{b}) = 1$
and
(II) $(a \cdot b) \cdot (\bar{a} + \bar{b}) = 0$

In class we proved statement (II) above. Here you will prove statement (I). In your proofs, you *cannot use the Duality Theorem.*

(b) Prove statement (I), please use the Associativity Theorem in your proof.

(c) Prove statement (I) **without** using the Associativity Theorem in your proof.

(d) Use DeMorgan's Theorem and other axioms and theorems to find the complements of: (i) $(a \cdot \bar{b}) + (\bar{a} \cdot b)$, (ii) $(a + b + c + d)$, and (iii) $a + (\bar{a} \cdot b \cdot c)$.
Please justify *every step* in your derivations using the axioms, lemmas and theorems from class.