Homework #5

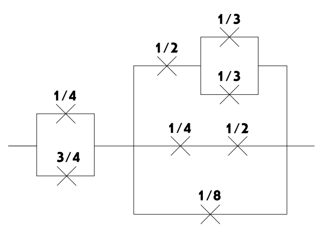
Due Thursday, June 7, 2018, at 2:30 PM

Collaboration and discussions are not allowed on Problem 1
and are allowed and encouraged on Problem 2

1. **Stochastic Switching Circuits** (*Collaboration is not allowed on this Problem*)
In this problem we study the synthesis and analysis of stochastic switching circuits. In particular, we consider simple series-parallel circuits, series-parallel circuits and general circuits.

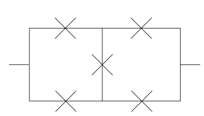
   (a) In class we discussed an algorithm (called the B-algorithm) for synthesizing simple series-parallel circuits for realizing probabilities of the form $a/2^n$, where $0 \leq a \leq 2^n$. Use the B-algorithm to synthesize a simple series-parallel circuit that computes 115/512 using only 1/2-pswitches; draw the circuit. Draw a circuit that computes 397/512 using 1/2-pswitches. How are the two circuits you synthesized related? Show your work.

   (b) What is the probability that the following series-parallel circuit is closed? Show your work. Can you increase the probability that the following circuit is closed by exchanging the probabilities of two pswitches? Show your work.

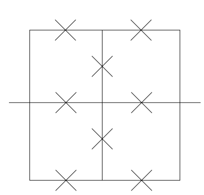


   (c) What is the probability that the following non-series-parallel circuits are closed? Assume that each pswitch is closed with probability 1/2. Show your work.

   i.



   ii.



2. **Linear Threshold (LT) Gates** (*Collaboration is allowed on this Problem*)
A linear threshold gate computes a Boolean function $f(X)$ as follows,

$$f(X) = sgn(F(X)) = sgn(w_0 + \sum_{i=1}^{n} w_i x_i)$$

where,

$$sgn(F(X)) = \begin{cases} 0 & \text{if } F(X) < 0 \\ 1 & \text{if } F(X) \geq 0 \end{cases}$$

We say that a Boolean function is in the class LT_1 if it can be computed by a single LT gate. An LT circuit is a circuit that consists of LT gates.

(a) Recall (from HW#4) that the Complete Quadratic (CQ) function is the Boolean function that consists of the XOR of all the $\binom{n}{2}$ possible AND's between pairs of inputs. Namely,

$$CQ(X) = (x_1 \cdot x_2) \oplus (x_1 \cdot x_3) \oplus \cdots \oplus (x_{n-1} \cdot x_n)$$

Draw a depth-2 *layered* LT circuit that computes $CQ(X)$ with six inputs, namely, $X \in \{0,1\}^6$. Please use as few LT gates as possible. We expect you to use the method (due to Muroga) that was described in class.

(b) We generalize linear threshold functions to accept more than two possible values in their inputs. Assume that every input x_i to a threshold function comes from an alphabet of size q, namely, $x_i \in \{0, 1, 2, \ldots, q-1\}$ (the binary case is when $q = 2$, namely, $x_i \in \{0,1\}$). For this generalization, we define the functions OR_q and AND_q as follows:

$$OR_q(x_1, x_2, \ldots, x_n) = \begin{cases} 0 & \text{if } x_i = 0 \text{ for all } 1 \leq i \leq n \\ 1 & \text{otherwise} \end{cases}$$

and

$$AND_q(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } x_i \neq 0 \text{ for all } 1 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$$

i. For which $q \geq 2$ is $OR_q \in LT_1$? Prove your claim.

ii. For which $q \geq 2$ is $AND_q \in LT_1$? Prove your claim.