IST 4
Information and Logic
HW3 will be returned today
Average is 24/28~\approx 85\%
### IST 4: Planned Schedule - Spring 2015

<table>
<thead>
<tr>
<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thr</th>
<th>Fri</th>
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- **T** = today
- **x** = hw#x out
- **x** = hw#x due
- **oh** = office hours
- **Mx** = MQx out
- **Mx** = MQx due
- **midterms**

**Important Dates:**
- HW#1 due (13th)
- HW#2 due (20th)
- HW#3 due (27th)
- HW#4 due (11th)
- HW#5 due (18th)
- HW#6 due (25th)
- HW#7 due (5th)

**Office Hours:**
- M2 midterms
Shannon

the last page of his MS thesis...
is the first page of logic design
A Symbolic Analysis of Relay and Switching Circuits

\[
\begin{array}{c|c|c|c|c}
   c_{k+1} & c_k & c_{j+1}c_j & c_{j+1}c_j1 & c_{j+1}c_j0 \\
   a_{k-1}a_{j+1}a_{j-1}a_1 & a_{k-1}b_{j+1}a_{j-1}b_1 & b_{k-1}b_{j+1}b_{j-1}b_1 \\
   \hline
   \text{Carried numbers} & \text{First number} & \text{Second number} \\
   \text{or} & \text{Sum} & \text{S} \\
   s_{k+1} & s_{j+1}+s_j & s_{j+1}+s_j & s_{j+1}+s_j0 \\
\end{array}
\]

Starting from the right, \( s_0 \) is one if \( a_0 \) is one and \( b_0 \) is zero or if \( a_0 \) is zero and \( b_0 \) one but not otherwise. Hence

\[ s_0 = a_0b_0 + a_0b_0 = a_0 \oplus b_0. \]

\( c_1 \) is one if both \( a_0 \) and \( b_0 \) are one but not otherwise:

\[ c_1 = a_0 \cdot b_0. \]

\( s_j \) is one if just one of \( a_j, b_j, c_j \) is one, or if all three are one:

\[ s_j = S_{1,3}(a_j, b_j, c_j), \quad j = 1, 2, \ldots k. \]

\( c_{j+1} \) is one if two or if three of these variables are one:

\[ c_{j+1} = S_{2,3}(a_j, b_j, c_j), \quad j = 1, 2, \ldots k. \]

Using the method of symmetric functions, and shifting down for \( s_j \) gives the circuits of Figure 35. Eliminating superfluous elements we arrive at Figure 36.

**References**

1. A complete bibliography of the literature of symbolic logic is given in the *Journal of Symbolic Logic*, volume I, number 4, December 1936. Those elementary parts of the theory that are useful in connection with relay circuits are well treated in the two following references.


Progress happens with the introduction of new languages.

Progress stops with illiteracy.
Illiteracy

Life and Brain
The operating principles of the two most important information systems, Life and the Human Brain, are still mysterious.

We need new languages and new mathematics.
Illiteracy

ideas for biological circuits
Learn from the “old masters”

- Reasoning
- Language
- Physics
Learn from the “old masters”

reasoning

language

Biology
Stochastic chemical reactions Biological

how is it special????
Composition is stochastic!
The challenge:
Analysis of stochastic behavior is not intuitive

However, synthesis is intuitive...
What should we do?

Idea:
*Synthesis of stochastic behavior*
A Symbolic Analysis of Relay and Switching Circuits

Claude E. Shannon**

Idea:
Synthesis of stochastic behavior

Relay on the edge is controlled by a 0-1 variable

The seed Idea: Relay on the edge is controlled by a Bernoulli (0-1) random variable
Idea: Synthesis of stochastic behavior

Closed with probability $1/2$

The seed Idea: Relay on the edge is controlled by a Bernoulli (0-1) random variable
Closed with probability ??

\[
\frac{1}{2} \times \frac{1}{2} = \frac{1}{2^2}
\]

Closed with probability ??

\[
1 - (1 - \frac{1}{2})^2
\]

both open

\[
= \frac{3}{2^2}
\]
Stochastic Circuits

first analysis...

of stochastic relay circuits
What is the probability that the following circuit is closed?

\[ \frac{3}{4} \times \frac{1}{2} \]
What is the probability that the following circuit is closed?

Notice that the probability next to a pswitch indicates the probability it is closed.
Method 1: Sub-block analysis

What is the probability that the following circuit is closed?

Probability it is open
\[
(1 - \frac{1}{2}) \times (1 - \frac{3}{4}) = \frac{1}{8}
\]

Probability it is closed
\[
1 - \frac{1}{8} = \frac{7}{8}
\]

Probability the circuit is closed
\[
\frac{7}{8} \times \frac{2}{3} = \frac{7}{12}
\]
Method 2: The Conditional Probability approach

What is the probability that the following circuit is closed?
Method 2: The Conditional Probability approach

What is the probability that the following circuit is closed?

$$Prob\{C\} = \frac{1}{2} Prob\{C|\text{selected switch is closed}\} + (1 - \frac{1}{2}) Prob\{C|\text{selected switch is open}\}$$

$$= \frac{1}{2} \times \frac{2}{3} + (1 - \frac{1}{2}) \times \frac{1}{2} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$
Stochastic Circuits

now synthesis...
Synthesis: Series-Parallel relay circuits are a Boolean algebra

Can compute any Boolean function

\[ a \cdot b \]

\[ a + b \]
Let \( p = \frac{a}{2^n} \) (The target probability)

With \( a \) an integer such that, \( 1 \leq a \leq 2^n \)

**Goal:**
Design a **series-parallel** stochastic switching circuit with 0.5-pswitches that computes (is closed with) a probability \( p \)
Let's compose and see what happens...

With **three** 0.5-pswitches can compute all $a/8$
Theorem (the expressive power of 0.5):

Let \( p = \frac{a}{2^n} \) (The target probability)

With \( a \) an integer such that, \( 1 \leq a \leq 2^n \)

Then there exists a (\textbf{simple}) series-parallel stochastic switching circuit with 0.5-pswitches, of size at most \( n \), that realizes \( p \)

Remarks:
\begin{itemize}
  \item Size \( n \) is \textbf{optimal} (for any circuit structure)
  \item There is an \textbf{efficient algorithm}
\end{itemize}
Series-Parallel Circuits

In series:

In parallel:
SIMPLE Series-Parallel Circuits

In series:

In parallel:
Is it a SIMPLE Series-Parallel Circuit?

No

Yes
What is the probability that the circuit is closed? \( \frac{1}{2} \) pswitches
What is the probability that the circuit is closed? $\frac{1}{2}$ pswitches

The construction algorithm (B-Algorithm):

Start from the right:

0 = pswitch in series
1 = pswitch in parallel

\[ \frac{11}{16} \]

.1011
The construction algorithm (B-Algorithm):

Start from the right:

0 = pswitch in series
1 = pswitch in parallel
Why Does the B-Algorithm Work?

**Proof:** By induction

Base case, 1 bit:

Assume true for \( n \) bits, prove for \( n+1 \) bits:

\( X \) has \( n \) bits

\[
\frac{1}{2} \times (0.X)_2 = (0.0X)_2
\]

\[
\frac{1}{2} \times 1 + \frac{1}{2} \times (0.X)_2 = (0.1X)_2
\]
Stochastic Circuits

duality
Recall Theorem 0:
Any identity that is true algebra, is also true
if $+$ and $.$ are interchanged, and $0$ and $1$ are interchanged.

- **A1. Identities:**
  \[ a + 0 = a \quad \text{and} \quad a \cdot 1 = a \]

- **A2. Complements:**
  \[ a + \bar{a} = 1 \quad \text{and} \quad a \cdot \bar{a} = 0 \]

- **A3. Commutativity:**
  \[ a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a \]

- **A4. Distributivity:**
  \[ a + b \cdot c = (a + b) \cdot (a + c) \quad \text{and} \quad a \cdot (b + c) = a \cdot b + a \cdot c \]
In Boolean algebra:

dual\{a + b\} \equiv a \cdot b

dual\{a \cdot b\} = a + b

Theorem 4 (DeMorgan):

\bar{a} + \bar{b} = (a \cdot b)

\bar{a} \cdot \bar{b} \equiv (a + b)

f(a, b) \equiv a + b

f(\bar{a}, \bar{b}) = \bar{a} + \bar{b}

\text{In Boolean general:}

dual\{f(\bar{a}, \bar{b})\} \equiv \bar{a} \cdot \bar{b}

\equiv (a + b)

\equiv f(a, b)

dual\{f(x_1, x_2, \ldots, x_n)\} = f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)
\[
dual\{f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)\} = \overline{f(x_1, x_2, \ldots, x_n)}
\]

**Example - Boolean functions:**

\[
f(x_1, x_2) = x_1 \cdot \overline{x}_2 + \overline{x}_1 \cdot x_2
\]

\[
dual[f(\overline{x}_1, \overline{x}_2)] = (\overline{x}_1 + x_2) \cdot (x_1 + \overline{x}_2)
\]

\[
= \overline{x}_1 \cdot \overline{x}_2 + x_1 \cdot x_2
\]

\[
= f(x_1, x_2)
\]
Dual circuits
How are dual stochastic circuits related?
In Boolean algebra:

\[ \text{dual}[f(\bar{X})] = \overline{f(X)} \]

**Theorem:** In stochastic series-parallel circuits

\[ \text{Prob}\{\text{dual}[C(\bar{X})]\} = 1 - \text{Prob}\{C(X)\} \]

\[ \bar{x} = 1 - x \]
Theorem: In stochastic series-parallel circuits

\[ \text{Prob}\{\text{dual}[C(\bar{X})]\} = 1 - \text{Prob}\{C(X)\} \]

The circuit:

\[ \frac{1}{2} \]

\[ \frac{2}{3} \]

\[ \frac{3}{4} \]

The dual(comp):

\[ \frac{1}{2} \times 1 + \frac{2}{3} \times \frac{1}{8} = \frac{5}{12} \]
Why Does the B-Algorithm Work?

Theorem (the expressive power of 0.5):

Let \( p = \frac{a}{2^n} \) (The target probability)

With \( a \) an integer such that, \( 1 \leq a \leq 2^n \)

Then there exists a (simple) series-parallel stochastic switching circuit with 0.5-pswitches, of size at most \( n \), that realizes \( p \)
**Why Does the B-Algorithm Work?**

**Proof:** By duality and induction...

0.5 = 1 - 0.5

With two pswitches:

Three pswitches:

- add one pswitch in series
- create the dual circuits
Stochastic Circuits

back to biology
Computing with stochastic behavior

Why is it useful?
How???

Can we program independent objects to behave in a 'coordinated way' without communications?

Access to a global variable – "ditch day"

Choose the behavior with probability

From probabilities to approximate quantities...
Creating a **global behavior** (dosage of insulin) using a large collection of independent cells that react to **global variables** (glucose level)

Access to a global variable – “ditch day”

Choose the behavior with probability

From probabilities to approximate quantities...
We created theoretical methods for the synthesis of stochastic circuits...

Can we map it to biology?
Constructing a Stochastic DNA Switch

Idea ???

\[
\begin{align*}
\text{Smiley} + \text{Red Face} & \rightarrow \text{Sad Face} \\
\text{Smiley} + \text{Happy Face} & \rightarrow \text{Happy Face}
\end{align*}
\]
Constructing a Stochastic DNA Switch

Idea ????

The probability distribution of the output is determined by the concentrations of
Experimental Results

a Stochastic DNA Switch

~2.9% error from manual pipetting
Toehold Mediated Branch Migration

Toehold Mediated Branch Migration

Toehold Mediated Branch Migration

Toehold Mediated Branch Migration

Toehold Mediated Branch Migration and Strand Displacement

Our building block = short DNA strands
It is **not** a connectivity model.
It is a **flow** model...
It is not a connectivity model
It is a flow model...

Implementation of stochastic networks with DNA strands inspired a new model of stochastic flow networks
Stochastic Circuits

flow and feedback
The basic building block in a flow model is a splitter.
Flow Model - Example
Flow Model - Example

\[
\begin{align*}
&\frac{1}{2} \\
&\frac{1}{2} \\
&\frac{2}{4} \\
\end{align*}
\]
It is not a **connectivity** model
It is a **flow** model...

New feature???

Feedback
Stochastic Flow with Feedback

What are $a$ and $b$?

$a = b = 1/2$

$p + q = 1$

HW#1
Fair from Biased

Von Neumann
1903–1957
Feedback Helps!!!

With $\frac{1}{2}$ splitters can compute:

\[ p = \frac{a}{2^n} \quad \text{without feedback} \]

\[ p = \frac{a}{b} \quad \text{with feedback} \]
Feedback Helps!!!

**Theorem:** Given a rational fraction $a/b$ with $2^{n-1} < b \leq 2^n$ and $a$ and $b$ are relatively prime numbers then $a/b$ can be realized by a circuit with $n$ $\frac{1}{2}$-splitters (3n arbitrary) size $n$ is optimal
Stochastic logic design:

- Expressive power and constructions
- Robustness

Experimental molecular circuits

Generalizations:
- Stochastic flow networks
- Min-Max networks
- Random-walk networks
It is All About People

Dan Wilhelm
The B-Algorithm
Experimental study with DNA strands

Lulu Qian

David Lee

Robustness
Stochastic flow networks

Po-Ling Loh
Hongchao Zhou
Ho-Lin Chen

Min-max
Random-walk networks

General expressive power