Homework # 4
Due Wednesday, February 22, 2006, at 1:30 PM
Collaboration is allowed and encouraged

1. Generalizing the Non-layered Construction
A symmetric function \( f(x_1, x_2, \cdots, x_n) \) can be described by a vector of length \( n + 1 \) that we call the symmetric function table and denote it by \( V(f) = (v_0, v_1, \cdots v_n) \), where

\[
v_i = \begin{cases} 
1 & \text{if for } |X| = i \quad f(X) = 1 \\
0 & \text{otherwise}
\end{cases}
\]

Let \( |V(f)| \) be the number of 1’s in \( V(f) \). In class we described a method for constructing a depth-2 non-layered LT circuit with \( |V(f)| + 1 \) gates.

(a) Let \( V(f) = \{011001010110\} \) be the symmetric function table of \( f \) (a function of 11 variables). Using the approach from class we can construct a depth-2 non-layered LT circuit with 7 gates (\( |V(f)| = 6 \)). However, note that \( V(f) \) has only 4 intervals of 1’s. Show how to implement \( f \) with a depth-2 non-layered LT circuit with 5 gates.

(b) Prove the general result. Namely, assume that you are given a symmetric function \( f \) with \( V(f) \) consisting of \( k \) intervals of 1’s. Prove that \( f \) can be implemented by a depth-2 non-layered circuit with \( k + 1 \) gates.

Strong hint: Look at the paper “Linear-Input Logic”, by R. C. Minnick, that is posted on the class web site.

2. Computing the Spectrum
Let \( f_1(x_1, x_2) = x_1 \land x_2 \) (AND function of two variables) and \( f_2(x_1, x_2) = x_1 \lor x_2 \) (OR function of two variables). In class we proved that AND and OR have the following polynomial representation.

\[
f_1(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)
\]

\[
f_2(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1x_2)
\]
(a) Derive the polynomial representation of \( f_3(x_1, x_2, x_3) = x_1 \land (x_2 \lor x_3) \).

(b) Derive the polynomial representations of:

\[
\text{AND}(x_1, x_2, x_3) = x_1 \land x_2 \land x_3
\]

\[
\text{OR}(x_1, x_2, x_3) = x_1 \lor x_2 \lor x_3
\]

(c) In general, for an arbitrary \( n \), compute the coefficients of the polynomial representations of \( \text{AND}(x_1, x_2, \ldots, x_n) \) and \( \text{OR}(x_1, x_2, \ldots, x_n) \) (note that the coefficients are functions of \( n \)).

3. The Spectrum of Symmetric Functions

A Boolean function is symmetric if and only if it is a function of the number of 1’s in the input. For example, PARITY, AND and OR are symmetric functions. Notice that the degree of their polynomial representation is \( n \).

(a) Prove that the degree of the polynomial representation of an arbitrary symmetric function \( f \) of \( n \) variables is at least \( \lceil n/2 \rceil \).

(b) For every \( n \) odd, find a symmetric function with \( n \) variables such that the degree of its polynomial representation is \( n - 1 \).

(c) Extra credit (15\%): Find an infinite sequence of even numbers, such that for every number \( n \) in the sequence there is a symmetric function with \( n \) variables such that the degree of its polynomial representation is \( n - 1 \).

4. The \( \{0, 1\} \) Representation

In class we proved that every Boolean function \( f(X) \in \{1, -1\}, X \in \{1, -1\}^n \), can be uniquely expressed as a polynomial with rational coefficients. The coefficients of the polynomial representation can be computed using the Sylvester-type Hadamard matrix.

In this problem we assume that we use \( \{0, 1\} \), namely a Boolean function \( f(X) \in \{0, 1\} \) is defined using \( X \in \{0, 1\}^n \) and study the corresponding polynomial representation that we call the \( \{0, 1\} \)-polynomial representation. For example, the \( \{0, 1\} \)-polynomial representation of \( \text{AND}(x_1, x_2) \) is \( x_1 x_2 \).

(a) Derive the \( \{0, 1\} \)-polynomial representation of \( \text{OR}(x_1, x_2, x_3) \) and \( \text{XOR}(x_1, x_2, x_3) \).

(b) In general, prove that the \( \{0, 1\} \)-polynomial representation is unique.

(c) Computing the coefficients: What is the transformation matrix from the function to the coefficients of the \( \{0, 1\} \)-polynomial representation?

Medium hint: derive the recursive definition of the transformation matrix.