CNS 188a Overview

- Boolean algebra as an axiomatic system
- Boolean functions and their representations using Boolean formulas and spectral methods
- Implementing Boolean functions with relay circuits, circuits of AON (AND, OR, NOT) gates and LT (Linear Threshold) gates
- Analyzing the complexity (size and depth) of circuits
- Relations (as opposed to functions) and their implementation in circuits
- Feedback and convergence in LT circuits

Emil Post
1897-1954

Compositions of Boolean functions
Universal Algebra
Linear Threshold
Some Adjustments

Linear Threshold (LT) gate

\[ f : \{-1, 1\}^n \rightarrow \{-1, 1\} \]

\[ a \rightarrow 2a - 1 \]

Threshold

\[ F(X) = -t + \sum_{i=1}^{n} w_i x_i \]

\[ f(X) = \text{sgn} \left( F(X) \right) = \begin{cases} -1 & \text{if } F(X) < 0 \\ 1 & \text{if } F(X) \geq 0 \end{cases} \]
Linear Threshold Example

The **AND** function of two variables:

\[ f(x_1, x_2) = \text{sgn}(-1 + x_1 + x_2) \]

<table>
<thead>
<tr>
<th>(x_1x_2)</th>
<th>(f(x_1, x_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1-1</td>
<td>-3</td>
</tr>
<tr>
<td>-11</td>
<td>-1</td>
</tr>
<tr>
<td>1-1</td>
<td>-1</td>
</tr>
<tr>
<td>11</td>
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</tr>
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Linear Threshold
New Ingredient

\[ f(X) = \text{sgn}(F(X)) \]

\[ F(X) = -t + \sum_{i=1}^{n} w_i x_i \]

A memory nose

Remembers the last \( f(X) \)
Feedback Networks

Example

The state of the network: the vector that corresponds to the states (noses...) of the gates
Feedback Networks
Example - state
Feedback Networks

Example - gate numbering

Label the gates
Feedback Networks
Example

```
  1 -1 0 1
-1 0 1
  2 -1 0 1
```
Feedback Networks

Example

\[\begin{array}{cccc}
-1 & 0 & -1 & 1 \\
-1 & 0 & 1 & 2 \\
\end{array}\]
Feedback Networks
Example

-1 -1 0 0 -1 1

is a stable state

-11 is a stable state
Feedback Networks
Example

-1 0 1

-1 0 1

1 2
Feedback Networks

Example
Feedback Networks

Example
Feedback Networks

Example

1-1 is a stable state
Feedback Networks
Example

State transition diagram (state space)

Q: Is -1-1 a stable state?
Feedback Networks
Example

Answer: No
Feedback Networks

Example

![Diagram of Feedback Networks Example]
Feedback Networks

Example
Feedback Networks

Example
Feedback Networks
Example

Stable states

Q: why care about stable states?
Feedback Networks
Computing with Dynamics

Stable states

Input: initial state

Feedback Network

Output: stable state
Feedback Networks
Computing with Dynamics

Stable states

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Feedback Network

Output: stable state
Feedback Networks
Computing with Dynamics

Input: initial state

Associative Memory
“The Leibniz-Boole Machine”

Feedback Network

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Output: stable state
Feedback Networks
Hopfield Model (Caltech 1982)

\[
\begin{array}{c}
1 \\
-1 \quad 0 \quad 1 \\
-1 \\
\end{array}
\quad \quad
\begin{array}{c}
2 \\
-1 \quad 0 \quad 1 \\
-1 \\
\end{array}
\]

\[i = \text{node } i\]
\[\theta = \text{threshold } t_i\]
\[1 = \text{state } v_i\]
\[-1 = \text{weight of edge } (i,j)\]
Feedback Networks

**Definition: feedback network**
- A directed graph with nodes and directed edges
- Every node $i$ has a state $v_i$
- Every node $i$ has a threshold $t_i$
- Every directed edge $(i,j)$ has a weight $w_{ij}$
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Feedback Networks

Computation in Each Node

\[ w_{j,i} \]

A node computes an LT function

\[ F_i(t) = \sum_{j=1}^{n} w_{j,i} v_j - t_i \]

The new state

\[ v_i(t + 1) = \text{sgn}(F_i(t)) \]
Feedback Networks
The Graph Description

Definition: feedback network
• A directed graph with nodes and directed edges
• Every node $i$ has a state $v_i$
• Every node $i$ has a threshold $t_i$
• Every directed edge $(i,j)$ has a weight $w_{i,j}$
Feedback Networks
The Vector/Matrix Description

An $n$ node feedback network can be specified by:

- $W$ an $n \times n$ matrix of weights
- $T$ an $n$ vector of thresholds
- $V$ an $n$ vector of states

$$N = (W, T)$$

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$T = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$$
**The Matrix Description**

**Example**

An *n* node feedback network can be specified by:

- $W$ an $n \times n$ matrix of weights
- $T$ an $n$ vector of thresholds
- $V$ an $n$ vector of states

\[
N = (W, T)
\]

**Diagram:**

![Diagram of a feedback network](image)

- $W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
- $T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
The Matrix Description

Computation

Computation in $N = (W, T)$

\[
F_i(t) = \sum_{j=1}^{n} w_{j,i} v_j - t_i
\]

by column

\[
F(t) = W^T \cdot V - T
\]

\[
V(t + 1) = \text{sgn}(F(t))
\]

\[
V = \begin{bmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_n
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
  t_1 \\
  t_2 \\
  \vdots \\
  t_n
\end{bmatrix}
\]
The Matrix Description

Example

Computation in $N = (W, T)$

$$F(t) = W^T \cdot V - T$$

$$
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
= \begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
$$

$$
= \begin{bmatrix}
-v_2 \\
-v_1
\end{bmatrix}
$$

Q: when do the nodes compute?
Modes of Operation

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Serial mode: one node at a time (arbitrary order)
Modes of Operation

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**Serial mode:** one node at a time (arbitrary order)
Modes of Operation

Q: when do the nodes compute?

Serial mode: one node at a time (arbitrary order)

Fully-Parallel mode: all nodes at the same time
Q: Are those stable states for serial or fully-parallel modes?

A: A state $V$ is stable iff (it is independent of the mode):

$$V = sgn \left( W^T \cdot V - T \right)$$

All the nodes must be happy.

Stable states

\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= 
\begin{bmatrix}
-v_2 \\
-v_1
\end{bmatrix}
\]

stable state $\iff v_1 = -v_2$
Example 1
Serial Mode - Symmetric Weight Matrix

The state space:

stable states

11 -11
1-1 -1-1

$W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Example 2
Fully-Parallel Mode – Symmetric Weight Matrix

Q: stable states in FP-mode?

Stable states are (same as in serial): -11 and 1-1

Q: what happens in the other states?

\[ W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \]

\[ T = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
Example 2
Fully-Parallel Mode - Symmetric Weight Matrix

Q: what happens in the other states?
A: start with 11

\[
sgn \left( \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}
\]

\[
sgn \left( \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

It's a cycle!

\[
W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}
\]

\[
T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
Example 2
Fully-Parallel Mode – Symmetric Weight Matrix

The state space:

stable states

cycle of length 2

$W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Example 3
Fully-Parallel Mode
Antisymmetric Weight Matrix

Q: how does the state space look?

\[ W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]
\[ T = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
Example 3
Fully-Parallel Mode – Antisymmetric Weight Matrix

Q: how does the state space look?

\[
\begin{align*}
T &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
W &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
sgn\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
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sgn\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{align*}
\]

cycle of length 4
Example 3
Fully-Parallel Mode - Antisymmetric Weight Matrix

The state space:

$$T = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

cycle of length 4
The Three Cases

Cycle lengths

Example #

<table>
<thead>
<tr>
<th>mode</th>
<th>symmetric</th>
<th>antisymmetric</th>
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</thead>
<tbody>
<tr>
<td>serial</td>
<td>1</td>
<td>?</td>
</tr>
<tr>
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The Three Cases

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Example #

1. Hopfield 1982
2. Goles 1985
3. Goles 1986
Proof Ideas

*Example #*

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The proofs of these three results use the concept of an **energy function**

For the serial mode:

\[ E(t) = V^T(t) \cdot W \cdot V(t) - 2V^T T \]

Show that:

\[ E(t + 1) - E(t) \geq 0 \]

Namely, stable states are local max of the energy \( E \)
 Proof Ideas

Example 1

The proofs of these three results use the concept of an energy function.

For the serial mode:

\[ E(t) = V^T(t) \cdot W \cdot V(t) - 2V^T T \]

Show that:

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Namely, stable states are local max of the energy \( E \)

\[
W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}
\]

\[
T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
E = [v_1, v_2] \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2v_1v_2
\]
Proof Ideas
Example 1

Stable states are local max of the energy $E$

$$E = [v_1, v_2] \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2v_1v_2$$
### Questions on Convergence

#### Cycle lengths

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#### Example #

1. Hopfield 1982
2. Goles 1985
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**Q1:** Are the three cases “distinct”?

**Q2:** Elementary proof? (wo/energy)

**Q3:** Other “interesting” cases?