CNS

Scored

188
100 Wilt Chamberlain, Philadelphia, 3/2/1962
81 Kobe Bryant, LA Lakers, 1/22/2006
78 Wilt Chamberlain, Philadelphia, 12/8/1961
73 David Thompson, Denver, 4/9/1978
73 Wilt Chamberlain, San Francisco, 11/16/1962
73 Wilt Chamberlain, Philadelphia, 1/13/1962
71 David Robinson, San Antonio, 4/24/1994
71 Elgin Baylor, LA Lakers, 11/15/1960
70 Wilt Chamberlain, San Francisco, 3/10/1963
CNS 188a Overview

- Boolean algebra as an axiomatic system
- Boolean functions and their representations using Boolean formulas and spectral methods
- Implementing Boolean functions with relay circuits, circuits of AON (AND, OR, NOT) gates and LT (Linear Threshold) gates
- Analyzing the complexity (size and depth) of circuits
- Relations (as opposed to functions) and their implementation in circuits
- Feedback and convergence in LT circuits
**XOR of Three Variables with AON**

\[ a \oplus b \oplus c = \overline{a} \cdot \overline{b} \cdot c + a \cdot b \cdot \overline{c} + a \cdot \overline{b} \cdot \overline{c} + a \cdot b \cdot c \]

- **Depth = 2**
- **Size = 5**

*is the complement*
XOR of Four Variables with AON

Depth = 2
Size = 9

4 ANDs for $|X| = 1$

4 ANDs for $|X| = 3$

Can we do better?
No    Yes

Size 9 optimal for depth 2

How?
Size 8 XOR AON Circuit of Four Variables

Size 5

\[ \text{XOR}(x,y,z) \]

Size 3

\[ \text{XOR}(x,y) \]

\[ \text{XOR}(a,b,c,d) \]

Size 9 optimal for depth 2

Size 8 is Optimal!
Neuron - Neural Gate

Linear Threshold (LT) gate

\[
x_i \in \{0, 1\}
\]

\[
F(X) = w_0 + \sum_{i=1}^{n} w_i x_i
\]

\[
f(X) = \text{sgn} (F(X)) = \begin{cases} 
0 & \text{if } F(X) < 0 \\
1 & \text{if } F(X) \geq 0
\end{cases}
\]
Can We Compute an AND Function with an LT Gate?

\[ F(x_1, x_2) = -2 + x_1 + x_2 \]

\[ F(X) = w_0 + \sum_{i=1}^{n} w_i x_i \]

<table>
<thead>
<tr>
<th>( x_1 ) ( x_2 )</th>
<th>( F(X) )</th>
<th>( f(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0   0</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>0   1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>1   0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>1   1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Can We Compute an OR Function with an LT Gate?

$OR(x_1, x_2)$

$F(x_1, x_2) = -1 + x_1 + x_2$

$F(X) = w_0 + \sum_{i=1}^{n} w_i x_i$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$F(X)$</th>
<th>$f(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Can We Compute a NOT with an LT Gate?

\[ f(x_1) = \text{sgn}(-2x_1 + 1) \]

Can we compute \( \text{NOT} \) without \( \text{sgn} \)?

\[ \bar{x}_1 = \text{NOT}(x_1) = 1 - x_1 \]
More Variables for AND?

\[ F(x_1, x_2) = -2 + x_1 + x_2 \]

\[ F(x_1, x_2, \ldots, x_n) = -n + x_1 + x_2 + \cdots + x_n \]

\[ |X| = \sum_{i=1}^{n} x_i \]

\[ -n + |X| \]

\[ F(X) = \begin{cases} 0 & \text{if } |X| = n \\ < 0 & \text{if } |X| < n \end{cases} \]

Hence \( \text{sgn}(F(X)) \) is an AND
More Variables for OR?

\[ F(x_1, x_2) = -1 + x_1 + x_2 \]

\[ F(x_1, x_2, \ldots, x_n) = -1 + x_1 + x_2 + \cdots + x_n \]

\[ |X| = \sum_{i=1}^{n} x_i \]

\[ -1 + |X| \]

\[ F(X) = \begin{cases} 
-1 & \text{if } |X| = 0 \\
\geq 0 & \text{if } |X| \geq 1 
\end{cases} \]

Hence, \( \text{sgn}(F(X)) \) is an OR
AON vs. LT Circuits

**Theorem (LT/AON):**

LT circuits are at least as powerful as AON circuits:

(i) **Size** of an AON circuit for a Boolean function \( f \) is greater/equal to the size of an optimal LT circuit for the function \( f \).

(ii) **Depth** of an AON circuit for a Boolean function \( f \) is greater/equal to the depth of an optimal LT circuit for the function \( f \).

**Proof:**

By the previous reductions: AND, OR and NOT can be implemented by a single LT gate.
LT gates are MORE Powerful

XOR of three variables with AON
- Depth = 2
- Size = 5

XOR of four variables with AON
- Depth = 2
- Size = 9

Size is optimal for depth 2

XOR of four variables with AON, size 8 is optimal for any depth
LT gates are MORE Powerful

Size 5 is optimal for AON depth 2

Size 4 LT depth 2

\[\text{XOR}(x_1, x_2, x_3)\]
LT gates are MORE Powerful

**LT-1 = LT layered, inputs go to first layer only**

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

| $|X|$ | $A$ | $B$ | $C$ | $A+B+C$ | $-2+A+B+C$ |
|-----|-----|-----|-----|---------|------------|
| 0   | 0   | 1   | 0   | 1       | -1         |
| 1   | 1   | 1   | 0   | 2       | 0          |
| 2   | 1   | 0   | 0   | 1       | -1         |
| 3   | 1   | 0   | 1   | 2       | 0          |

$LT-l = LT$ layered, inputs go to first layer only

Can take the sgn or add 1

$XOR(x_1, x_2, x_3)$
LT gates are EVEN MORE Powerful

LT-nl = LT non-layered, inputs go to any layer

Can take the sgn or add 1
**XOR Function: Size of LT vs. AON in Depth 2**

<table>
<thead>
<tr>
<th></th>
<th>$XOR(x_1, x_2, x_3)$</th>
<th>$XOR(x_1, x_2, \ldots, x_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AON</td>
<td>5</td>
<td>$2^{n-1} + 1^*$</td>
</tr>
<tr>
<td>LT-I</td>
<td>4</td>
<td>$n + 1^*$</td>
</tr>
<tr>
<td>LT-nl</td>
<td>2</td>
<td>$\left\lceil \frac{n}{2} \right\rceil + 1^*$</td>
</tr>
</tbody>
</table>

* = it is optimal

Exponential gap in size
XOR with a Single LT Gate

Is it possible to compute $\text{XOR}(x_1, x_2)$ with a single LT element?

Answer: NO

Proof: By contradiction, assume it is possible and reach a contradiction.

\[
\text{XOR}(x_1, x_2) = \text{sgn}(w_0 + w_1 x_1 + w_2 x_2)
\]

- $x_1 = 0, x_2 = 0 \implies w_0 < 0$
- $x_1 = 1, x_2 = 0 \implies w_0 + w_1 \geq 0$
- $x_1 = 0, x_2 = 1 \implies w_0 + w_2 \geq 0$
- $x_1 = 1, x_2 = 1 \implies w_0 + w_1 + w_2 < 0$

Contradiction:

$2w_0 + w_1 + w_2 < 0$

$2w_0 + w_1 + w_2 \geq 0$
History on LT

Warren McCulloch
1899 - 1969

Walter Pitts
1923 - 1969

Neurophysiologist, MD

Logician, Autodidact

Computing with LT (neural) gates: connection between circuits and neural networks, 1943

Warren McCulloch arrived in early 1942 to the University of Chicago, invited Pitts, who was still homeless, to live with his family.

In the evenings McCulloch and Pitts collaborated. Pitts was familiar with the work of Gottfried Leibniz on computing and they considered the question of whether the nervous system could be considered a kind of universal computing device as described by Leibniz.

This led to their 1943 seminal neural networks paper: *A Logical Calculus of Ideas Immanent in Nervous Activity.*
Symmetric Functions

AND, OR and XOR are symmetric Boolean functions

Permuting the inputs does not change the output

\[ XOR(x_1, x_2, x_3) = XOR(x_2, x_1, x_3) = XOR(x_2, x_3, x_1) \]

Definition: A Boolean function \( f \) is symmetric if

\[ f(X) = f(\pi(X)) \]

for an arbitrary permutation \( \pi \)
Questions on Symmetric Functions

AND, OR and XOR are symmetric Boolean functions

Q1: How many symmetric Boolean functions of n variables?

Q2: What are the symmetric functions that can be realized by a single LT gate?
Symmetric Functions

**Definition:** A Boolean function $f$ is symmetric if

$$f(X) = f(\pi(X))$$

for an arbitrary permutation $\pi$

**Theorem:** A Boolean function $f(X)$ is symmetric if and only if it is a function of the number of 1's in $X$, namely $|X|$
Symmetric Functions

**Theorem:** A Boolean function \( f(X) \) is symmetric if and only if it is a function of the number of 1's in \( X \), namely \( |X| \)

**Proof:**

- **Given:** \( |X_1| = |X_2| \)
- **Need to prove:** \( f(X_1) = f(X_2) \)

\[
|X| = |\pi(X)| \quad \forall \pi, \forall X
\]

\[
f(X) = f(\pi(X)) \quad \forall \pi, \forall X
\]
**Theorem:** A Boolean function $f(X)$ is symmetric if and only if it is a function of the number of 1's in $X$, namely $|X|$

**Proof:**

- If $f$ is symmetric, then $f$ is a function of $|X|$.

Given: $f(X) = f(\pi(X))$ $\forall \pi, \forall X$

Need to prove: $|X_1| = |X_2|$ $\Rightarrow f(X_1) = f(X_2)$

However, $|X_1| = |X_2|$ $\Rightarrow \exists \pi, X_2 = \pi(X_1)$

Hence, $f(X_1) = f(\pi(X_1)) = f(X_2)$
Questions on Symmetric Functions

Q1: How many symmetric Boolean functions of n variables?

Q2: What are the symmetric functions that can be realized by a single LT gate?

A1: Theorem: The number of symmetric functions of n variables is: \(2^n + 1\)
Number of Symmetric Functions

Theorem: The number of symmetric functions of \( n \) variables is: \( 2^{n+1} \)

Proof:

\[
\begin{array}{c|c}
|X| & f(X) \\
0 & * \\
1 & * \\
2 & * \\
\vdots & \vdots \\
n & * \\
\end{array}
\]

* can be 0 or 1

\( 2^{n+1} \) functions
Questions on Symmetric Functions

Q1: How many symmetric Boolean functions of $n$ variables?

Q2: What are the symmetric functions that can be realized by a single LT gate?

Definitions:

(1) $\text{SYM} = \text{the class of Boolean symmetric functions}$

(2) $\text{LT}_1 = \text{the class of Boolean functions that can be realized by a single LT gate.}$

Q2: How is $\text{SYM}$ related to $\text{LT}_1$?
Questions on Symmetric Functions

Q2: How is $\text{SYM}$ related to $\text{LT}_1$?

Q3: Which class has more functions?
**SYM and LT$_1$**

**Theorem:**

\[ SYM \cap LT_1 = TH \]
**LT₁ Function that is not Symmetric**

\[ f(x_1, x_2) = x_1 \overline{x_2} \]

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(F(X))</th>
<th>(f(X))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ F(x_1, x_2) = -1 + x_1 - x_2 \]
SYM and $LT_1$

Q: What is $TH$?

$XOR$  

$AND$  

$OR$  

$f(x_1, x_2) = x_1 \bar{x}_2$
**Q: What is TH?**

**Definition:** A symmetric Boolean function is in TH if it has at most a single transition in the symmetric function table.

<table>
<thead>
<tr>
<th>$X$</th>
<th>AND</th>
<th>OR</th>
<th>XOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

= a transition

Not in TH
The Class TH

**Definition:** A symmetric Boolean function is in TH if it has at most a single transition in the symmetric function table.

\[
TH_i(x_1, x_2, \ldots, x_n) = \begin{cases} 
1 & \text{if } |X| \geq i \\
0 & \text{if } |X| < i 
\end{cases}
\]

- \(AND(X) = TH_n\)
- \(OR(X) = TH_1\)
- \(MAJ(X) = TH_{\left\lceil \frac{n}{2} \right\rceil}\)
The Class TH

\[ TH_i(x_1, x_2, \ldots, x_n) = \begin{cases} 
1 & \text{if } |X| \geq i \\
0 & \text{if } |X| < i 
\end{cases} \]

| |X| | TH_0 | TH_1 | TH_2 | TH_3 |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 |
| 2 | 1 | 1 | 1 | 0 | |
| 3 | 1 | 1 | 1 | 1 | 1 |
The Class TH

\[ TH_i(x_1, x_2, \ldots, x_n) = \begin{cases} 
1 & \text{if } |X| \geq i \\
0 & \text{if } |X| < i 
\end{cases} \]

\[ \overline{TH}_i = \{ f \mid \overline{f} = TH_i \} \]

\[ TH = \{ TH_i, \overline{TH}_i \mid 0 \leq i \leq n \} \]

| |X| | TH_0 | TH_1 | TH_2 | TH_3 | \overline{TH}_0 | \overline{TH}_1 | \overline{TH}_2 | \overline{TH}_3 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 3 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
The Class TH

\[ TH = \{ TH_i, \overline{TH_i} \mid 0 \leq i \leq n \} \]

**Q:** what is \(|TH|\)?

**A:** \(2n+2\)

| \(|X|\) | \(TH_0\) | \(TH_1\) | \(TH_2\) | \(TH_3\) | \(TH_0\) | \(TH_1\) | \(TH_2\) | \(TH_3\) |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0     | 1      | 0      | 0      | 0      | 0      | 1      | 1      | 1      |
| 1     | 1      | 1      | 0      | 0      | 0      | 0      | 1      | 1      |
| 2     | 1      | 1      | 1      | 0      | 0      | 0      | 0      | 1      |
| 3     | 1      | 1      | 1      | 1      | 0      | 0      | 0      | 0      |
We know that:

\[ TH \subset SYM \]

First, prove that:
\[ TH \subset LT_1 \]

Second, prove that:
\[ SYM \cap LT_1 = TH \]
Easy Part of the Proof

We know that:

\[ TH \subseteq SYM \]

\[ TH \subseteq LT_1 \]

\[ SYM \cap LT_1 \supseteq TH \]

First, prove that:

\[ TH_i(x_1, x_2, \ldots, x_n) = \begin{cases} 
1 & \text{if } |X| \geq i \\
0 & \text{if } |X| < i 
\end{cases} \]

\[ TH_i(x_1, x_2, \ldots, x_n) = sgn(-i + x_1 + x_2 + \cdots + x_n) \]

\[ = \begin{cases} 
1 & \text{if } |X| \geq i \\
0 & \text{if } |X| < i 
\end{cases} \]

\[ \overline{TH_i} = sgn((i - 1) - (x_1 + x_2 + \cdots + x_n)) \]
Harder Part of the Proof

We proved that: \( SYM \bigcap LT_1 \supseteq TH \)

Need to prove that: \( SYM \bigcap LT_1 = TH \)

Idea??

\( f \notin LT_1 \)
**Harder Part of the Proof**

**Idea:** proof by contradiction, assume $f \in LT_1$ exists

The function $f$ is in **SYM** and not **TH**, **wlog**:

| $|X|$ | 0 | 1 | 2 | 3 | * | * | * | * | * | n |
|-----|---|---|---|---|---|---|---|---|---|---|
| $f(X)$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |

**without loss of generality**
Harder Part of the Proof

Idea: proof by contradiction, assume $f \in LT_1$ exists

The function $f$ is in SYM and not TH, \textbf{wlog}:

| $|X|$ | 0 | 1 | 2 | 3 | * | * | * | * | * | n |
|------|---|---|---|---|---|---|---|---|---|---|
| $f(X)$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |

\textbf{without loss of generality}
Why WLOG?

**Idea:** proof by contradiction, assume $f \in LT_1$ exists

The function $f$ is in $\text{SYM}$ and not $\text{TH}$, **wlog**:

| $|X|$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|---|
| $f(X)$ | 0 | 0 | 1 | 0 | 1 | 1 |

$F(X) = w_0 + w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + w_5x_5$

$x_3 = 1 \quad x_4 = 0 \quad x_5 = 0$

$F(X) = w_0 + w_1x_1 + w_2x_2 + w_3 + 0 + 0$

$1 \leq |X| = x_1 + x_2 + 1 \leq 3$
Why WLOG?

Idea: proof by contradiction, assume \( f \in LT_1 \) exists

The function \( f \) is in \textbf{SYM} and not \textbf{TH}, \textbf{wlog}:

\[
\begin{array}{c|c|c|c|c|c|c}
|X| & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
f(X) & 0 & 0 & 1 & 0 & 1 & 1 \\
\end{array}
\]

\[
1 \leq |X| = x_1 + x_2 + 1 \leq 3
\]
Harder Part of the Proof

Idea: proof by contradiction, assume $f \in LT_1$ exists

WLOG $f$ is:

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>*</th>
<th>*</th>
<th>*</th>
<th>*</th>
<th>*</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(X)$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$\rightarrow$ XOR($x_1, x_2$)

$\rightarrow$ Q
We Proved the Theorem

Theorem: \( SYM \cap LT_1 = TH \)

2n+2 functions of SYM are in LT1, what about the rest of SYM?
More Layers

**Definition:**

\(LT_2\) = the class of Boolean functions that can be realized by a two layer circuit of \(LT\) gates, size is bounded by \(poly(n)\), for example \(n, n^2, n^3, \ldots, n^k\) where \(k\) is a constant independent of \(n\)

**Q:** How is \(SYM\) related to \(LT_2\)??

\(2n+2\) functions of \(SYM\) are in \(LT_1\), what about the rest of \(SYM\)?
How is SYM related to LT$_2$? 

| $|X|$ | $TH_1$ | $\overline{TH_2}$ | $TH_1+\overline{TH_2}-1$ |
|-----|--------|-------------------|--------------------------|
| 0   | 0      | 1                 | 0                        |
| 1   | 1      | 1                 | 1                        |
| 2   | 1      | 0                 | 0                        |